Reinforced Concrete Frame Element with Bond Interfaces. I: Displacement-Based, Force-Based, and Mixed Formulations

Suchart Limkatanyu[1] and Enrico Spacone[2]

Abstract: This is the first of two papers that presents the theory and applications of three different formulations of reinforced concrete frame elements with bond slip in the reinforcing bars. This paper presents the governing differential equations of the problem (strong form) and the three different element formulations (weak forms). The first is the displacement-based formulation, which is derived from the principle of stationary potential energy. The second is the force-based formulation, which is derived from the principle of stationary complementary energy. The third is the two-field mixed formulation, which is derived from the principle of stationary Hellinger–Reissner potential. The selection of the displacement and force shape functions for the different formulations is discussed. Tonti's diagrams are used to conveniently represent the equations that govern both the strong and the weak forms of the problem. This paper derives the general matrix equations of the three formulations. Implementation of the formulations in a general-purposed nonlinear structural analysis software and a set of applications are discussed in the companion paper.

DOI: 10.1061/(ASCE)0733-9445(2002)128:3(346)

CE Database keywords: Concrete; Reinforced; Frames; Bonding; Beams; Finite elements; Nonlinear analysis.

Introduction

The accurate modeling of the bond-slip effects is central in assessing the seismic performance of reinforced concrete (RC) frames under both monotonic and cyclic loads. Under the assumption of full bond between the concrete and the reinforcing bars, the stiffness and strength of RC structures are overestimated, as is the energy dissipated during the loading cycles. A number of experimental tests on RC subassemblages have shown the reduction in stiffness due to slip in the reinforcing bars above the foundations and in the beam–column connections. Bar pullouts, which may be experienced in older structures with insufficient bar anchorages or bar splices, drastically reduce the strength of the RC members. In order to realistically describe the behavior of RC structures in static and dynamic nonlinear structural analysis, the inclusion of the bond-slip effects is essential.

Several analytical models that include the bond-slip effects in the response of RC structures have been proposed in the published literature. These models are classified here into two groups: (1) solid two-dimensional or three-dimensional finite element models based on a continuum approach; (2) line one-dimensional (1D) frame elements based on the beam theory. Examples and applications of solid finite elements developed for the analysis of RC structures including the effects of bond slip are found in Ngo and Scordelis (1967), Nilson (1971), Kwak and Filippou (1995), and more recently, Lowes (1999). Analyses performed with solid elements provide greater details but are hampered by high computational efforts. Only structural components or small frames can be realistically studied using solid elements. For the nonlinear analysis of entire frames under monotonic and cyclic loads, frame elements capture the main characteristics of the structural response with reasonable computational costs. This work focuses on the development of a new family of 1D frame elements for RC members with bond slip.

Filippou and Issa (1988), and Filippou et al. (1999) proposed a so-called subelement frame element model, where the different sources of material nonlinearities (e.g., bending deformations, shear deformations, bond slip) are represented by separate subelements. Rubiano-Benavides (1998) proposed a fiber frame model with nonlinear rotational springs at the member ends that account for the bond slip. Monti and Spacone (2000) combine the force-based fiber beam element by Spacone et al. (1996) with the reinforcing bar element with bond slip by Monti et al. (1997). This last model is probably the most refined RC frame element with bond slip proposed to date. It relies on the simple idea that the total steel fiber strain at a cross section is the actual steel strain plus a strain equivalent to the bar slip. Although effective and precise, this model requires a complex and computationally demanding fiber state determination.

The RC frame elements with bond slip proposed in this work are different from those published to date in that different degrees of freedom are used for the concrete beam and for the bars with bond slip. In other words, the bond slip between the steel bar and the surrounding concrete is computed directly as the difference in the steel and concrete displacements at the bar level. Spacone and Limkatanyu (2000) have already successfully tested the general idea with the simple, displacement-based formulation. The result-

[1]Lecturer, Dept. of Civil Engineering, Prince of Songkla Univ., Hadyai, Thailand 90110.

[2]Associate Professor, Dept. of Civil, Environmental, and Architectural Engineering, Univ. of Colorado, Boulder, CO 80309-0428. E-mail: spacone@colorado.edu

Note. Associate Editor: Marc I. Hoit. Discussion open until August 1, 2002. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this paper was submitted for review and possible publication on February 21, 2001; approved on August 17, 2001. This paper is part of the ***Journal of Structural Engineering***, Vol. 128, No. 3, March 1, 2002. ©ASCE, ISSN 0733-9445/2002/3-346–355/$8.00+$.50 per page.

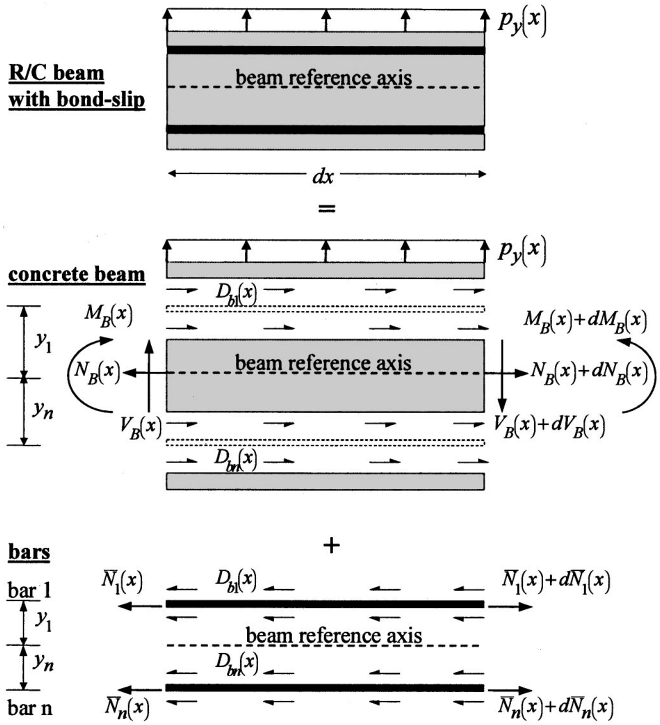

Fig. 1. Reinforced concrete beam element with bond slip: beam and bar components

ing model is computationally robust, but not very accurate. It is used hereafter as a reference model.

This paper presents the general theoretical framework of the displacement-based, force-based, and two-field mixed formulations of RC frame elements with bond slip in the steel bars. The beam section force–deformation relations are derived from the fiber section model. The derivation of the governing differential equations (strong form) of the RC frame element with bond interfaces is presented first. The three different element formulations (weak forms) are presented next and form the core of this paper. The displacement-based element derives from the principle of stationary potential energy functional. The force-based element derives from the principle of stationary complementary potential energy functional. The two-field mixed element derives from the principle of stationary Hellinger–Reissner (H–R) mixed functional. The Tonti's diagrams are used to concisely illustrate the governing equations of both the strong and the weak forms. A companion paper (Limkatanyu and Spacone 2002) discusses the implementation issues of the three different formulations in a general-purpose nonlinear structural analysis software and presents a set of applications and parametric studies.

Equations of Reinforced Concrete Frame Element with Bond Slip (Strong Form)

Equilibrium

The free body diagram of an infinitesimal segment dx of RC frame element with n bars with bond interfaces is shown in Fig. 1. Only bond stresses tangential to the bars are considered in this formulation. The dowel effect in the bars is neglected. Based on the small-deformation assumption, all of the equilibrium conditions are considered in the undeformed configuration. Axial equilibriums in the beam component and in the bar i lead to the following equations:

$$\frac{dN_B(x)}{dx}+\sum_{i=1}^{n} D_{\mathrm{bi}}(x)=0$$
$$\frac{dN_i(x)}{dx}-D_{\mathrm{bi}}(x)=0, \quad i=1,n \tag{1}$$

where $N_B(x)$ = the axial forces in the beam and $N_i(x)$ = the axial forces in the bar i, respectively. $D_{\mathrm{bi}}(x)$ = the bond interface force between the beam and bar i. Vertical equilibrium of the infinitesimal segment dx yields

$$\frac{dV_B(x)}{dx}-p_y(x)=0 \tag{2}$$

where $V_B(x)$ = the beam section shear force and $p_y(x)$ = the transverse distributed load. Finally, moment equilibrium yields

$$\frac{dM_B(x)}{dx}-V_B(x)-\sum_{i=1}^{n} y_i D_{\mathrm{bi}}(x)=0 \tag{3}$$

where $M_B(x)$ = the beam section bending moment, and y_i = the distance of bar i from the element reference axis (Fig. 1). This work follows the Euler–Bernoulli beam theory, thus the shear deformations are neglected. The shear force $V_B(x)$ is removed by combining Eqs. (2) and (3) to obtain

$$\frac{d^2M_B(x)}{dx^2}-p_y(x)-\sum_{i=1}^{n} y_i\frac{dD_{\mathrm{bi}}(x)}{dx}=0 \tag{4}$$

Eqs. (1) and (4) represent the governing equilibrium equations of the RC frame element with bond slip. Eqs. (1) and (4) can be written in the following matrix form:

$$\partial_B^T \mathbf{D}_B(x)-\partial_b^T \mathbf{D}_b(x)-\mathbf{p}(x)=\mathbf{0} \tag{5}$$

where $\mathbf{D}_B(x)=\{\bar{\mathbf{D}}(x):\bar{\bar{\mathbf{D}}}(x)\}^T$ = the element section forces, $\bar{\mathbf{D}}(x)=\{N_B(x)\ M_B(x)\}^T$ = the beam section forces, $\bar{\bar{\mathbf{D}}}(x)=\{N_1(x)\ldots N_n(x)\}^T$ = the bar forces, $\mathbf{D}_b(x)=\{D_{b1}(x)\ldots D_{bn}(x)\}^T$ = the bond section forces, and $\mathbf{p}(x)=\{0\ p_y(x)\ 0\ldots 0\}^T$ = the element force vector. ∂_B and ∂_b are differential operators defined in the Notation. It is important to point out that there are $2n+2$ internal force unknowns, while only $n+2$ equilibrium equations are available at any element section. Consequently, this system is internally statically indeterminate and the internal forces cannot be determined solely by the equilibrium conditions.

Compatibility

The element section deformation vector conjugate of $\mathbf{D}_B(x)$ is $\mathbf{d}_B(x)=\{\bar{\mathbf{d}}(x):\bar{\bar{\mathbf{d}}}(x)\}^T$, where $\bar{\mathbf{d}}(x)=\{\varepsilon_B\ (x)\kappa_B(x)\}^T$ are the beam section deformations (axial strain ε_B at reference axis and curvature κ_B), and $\bar{\bar{\mathbf{d}}}(x)=\{\varepsilon_1(x)\ldots\varepsilon_n(x)\}^T$ contains the axial strains of the n bars. The following displacements are defined at the element level: $\mathbf{u}(x)=\{\bar{\mathbf{u}}(x):\bar{\bar{\mathbf{u}}}(x)\}^T$ are the displacement fields along the element, where $\bar{\mathbf{u}}(x)=\{u_B(x)\ v_B(x)\}^T$ contains the beam axial and transverse displacements, respectively, and $\bar{\bar{\mathbf{u}}}(x)=\{u_i(x)\ldots u_n(x)\}^T$ contains the axial displacements of the n bars.

From the small deformation assumption, the element deformations are related to the element displacements through the follow-

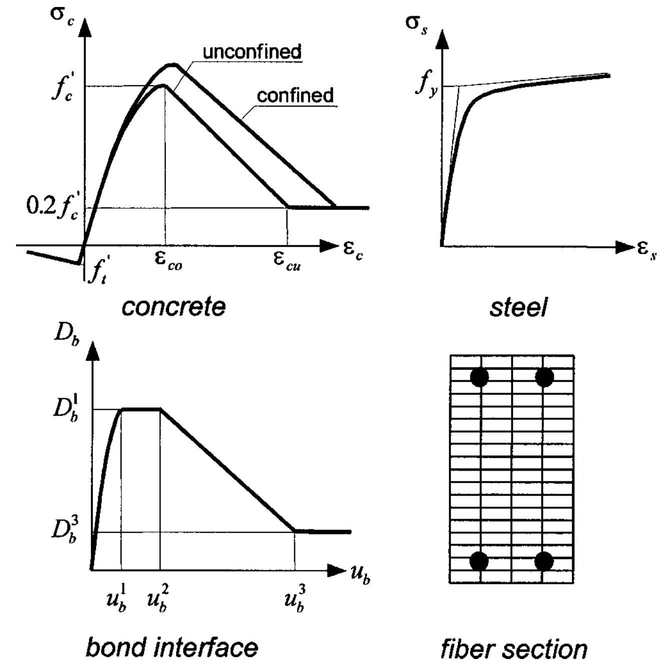

Fig. 2. Reinforced concrete beam element: section fiber discretization and material uniaxial laws

ing compatibility relations: $\varepsilon_B(x)=du_B(x)/dx$, $\kappa_B(x)=d^2v_B(x)/dx^2$, and $\varepsilon_i(x)=du_i(x)/dx$, which can be written in the following matrix form:

$$\mathbf{d}_B(x)=\partial_B\mathbf{u}(x) \tag{6}$$

The bond slips are determined by the following compatibility relation between the beam and the bar displacements:

$$u_{bi}(x)=u_i(x)-u_B(x)+y_i\frac{dv_B(x)}{dx} \tag{7}$$

where $u_{bi}(x)$ = the bond slip between the beam and bar i. If the bond deformation vector $\mathbf{d}_b(x)=\{u_{b1}(x)\ldots u_{bn}(x)\}^T$ is introduced, Eq. (7) can be written in the following matrix form:

$$\mathbf{d}_b(x)=\partial_b\mathbf{u}(x) \tag{8}$$

Force–Deformation Relations

The nonlinear nature of the proposed elements derives entirely from the nonlinear relation between the section forces $\mathbf{D}_B(x)$, $\mathbf{D}_b(x)$, and the section deformation $\mathbf{d}_B(x)$, $\mathbf{d}_b(x)$. In the proposed formulations, the fiber section model shown in Fig. 2 is used to derive the constitutive law $\mathbf{D}_B=\mathbf{D}_B(\mathbf{d}_B)$. The fiber model automatically accounts for the coupling between axial and bending responses. The Kent and Park (1971) law is used for the concrete fibers. The Menegotto and Pinto (1973) law is used for the bars. The explicit expression for the fiber beam section force deformation is given in Spacone et al. (1996). For the bond-slip constitutive relations $\mathbf{D}_b=\mathbf{D}_b(\mathbf{d}_b)$, the Eligehausen et al. (1983) law is used. All of these uniaxial laws are shown in Fig. 2.

In the following formulations, the section and bond-slip nonlinear laws are linearized according to the following forms:

$$\begin{aligned}\mathbf{D}_B(x)&=\mathbf{D}_B^0(x)+\Delta\mathbf{D}_B(x)=\mathbf{D}_B^0(x)+\mathbf{k}_B^0(x)\Delta\mathbf{d}_B(x)\\ \mathbf{D}_b(x)&=\mathbf{D}_b^0(x)+\Delta\mathbf{D}_b(x)=\mathbf{D}_b^0(x)+\mathbf{k}_b^0(x)\Delta\mathbf{d}_b(x)\end{aligned} \tag{9}$$

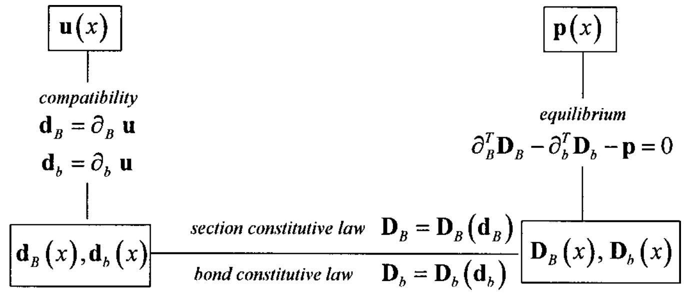

Fig. 3. Tonti's diagram for reinforced concrete beam with bond slip: differential equations (strong form)

where $\mathbf{D}_B^0(x)$, $\mathbf{D}_b^0(x)$, $\mathbf{k}_B^0(x)$, and $\mathbf{k}_b^0(x)$ = the section and bond force vectors and stiffness matrices at the initial point. The consistent inverse of Eq. (9) can be expressed in the following forms:

$$\begin{aligned}\mathbf{d}_B(x)&=\mathbf{d}_B^0(x)+\Delta\mathbf{d}_B(x)=\mathbf{d}_B^0(x)+\mathbf{f}_B^0(x)\Delta\mathbf{D}_B(x)\\ \mathbf{d}_b(x)&=\mathbf{d}_b^0(x)+\Delta\mathbf{d}_b(x)=\mathbf{d}_b^0(x)+\mathbf{f}_b^0(x)\Delta\mathbf{D}_b(x)\end{aligned} \tag{10}$$

where $\mathbf{d}_B^0(x)$, $\mathbf{d}_b^0(x)$, $\mathbf{f}_B^0(x)$, and $\mathbf{f}_b^0(x)$ = the section and bond deformation vectors and flexibility matrices at the initial point. In the above equations and throughout the paper, superscript 0 = the value of a vector or matrix at the initial point of the nonlinear scheme.

The compatibility, equilibrium, and constitutive equations for the RC frame element with bond slip presented above are conveniently represented in the classical Tonti's diagram of Fig. 3. This diagram is going to be modified for the different element formulations. Finally, for the sake of brevity, the transverse load $p_y(x)$ is omitted in the following derivations.

Finite Element Formulations (Weak Form)

Three different finite element formulations are presented in this paper: (1) a displacement-based formulation; (2) a hybrid force-based formulation; and (3) a mixed formulation.

Displacement-Based Formulation

The displacement-based formulation was originally presented in Spacone and Limkatanyu (2000) and is briefly summarized here. The element nodal displacements $\mathbf{U}$ serve as the primary element unknowns. The section displacements $\mathbf{u}(x)$ are related to the element nodal displacements through appropriate displacement shape functions. The section deformations $\mathbf{d}_B(x)$ and bond slip $\mathbf{d}_b(x)$ are determined through the beam compatibility Eq. (6) and the bond-interface compatibility Eq. (8), respectively. Therefore, the compatibility conditions are satisfied pointwise. On the other hand, the equilibrium conditions are only satisfied in the integral sense through the principle of stationary potential energy. The above steps are summarized in Tonti's diagram of Fig. 4 for the displacement-based element. If the total potential energy functional is defined as

$$\Pi_{\mathrm{PE}}[\mathbf{u}(x)]=\int_L\mathbf{u}^T(x)[\partial_B^T\mathbf{D}_B(x)-\partial_b^T\mathbf{D}_b(x)]dx \tag{11}$$

Equilibrium is found by imposing that the first variation of Π_{PE} be equal to zero

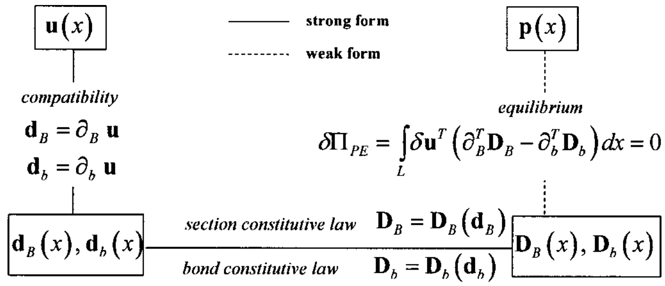

Fig. 4. Tonti's diagram for reinforced concrete beam with bond slip: displacement-based formulation

$$\delta\Pi_{\mathrm{PE}}[\delta\mathbf{u}(x)] = \int_L \delta\mathbf{u}^T(x)[\partial_B^T\mathbf{D}_B(x) - \partial_b^T\mathbf{D}_b(x)]dx = 0 \quad (12)$$

Upon substitution of the section linearized constitutive laws Eqs. (9) and (12) are written

$$\int_L \delta\mathbf{u}^T(x)\{\partial_B^T[\mathbf{D}_B^0(x) + k_B^0(x)\partial_B\Delta\mathbf{u}(x)] - \partial_b^T[\mathbf{D}_b^0(x) + k_b^0(x)\partial_b\Delta\mathbf{u}(x)]\}dx = 0 \quad (13)$$

where $\delta\mathbf{u}(x)$ = a compatible virtual displacement field. In order to move the differential operators ∂_B and ∂_b from the force vectors $\mathbf{D}_B(x)$ and $\mathbf{D}_b(x)$ to the virtual displacement fields $\delta\mathbf{u}(x)$, integration by parts is applied to Eq. (13), leading to

$$\int_L \begin{Bmatrix} \partial_B[\delta\mathbf{u}(x)] \\ \partial_b[\delta\mathbf{u}(x)] \end{Bmatrix}^T \begin{bmatrix} \mathbf{k}_B^0(x) & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_b^0(x) \end{bmatrix} \begin{Bmatrix} \partial_B[\Delta\mathbf{u}(x)] \\ \partial_b[\Delta\mathbf{u}(x)] \end{Bmatrix} dx = \delta\mathbf{U}^T\mathbf{P} - \int_L \{\partial_B[\delta\mathbf{u}^T(x)]\mathbf{D}_B^0(x) + \partial_b[\delta\mathbf{u}^T(x)]\mathbf{D}_b^0(x)\}dx \quad (14)$$

where $\delta\mathbf{U}^T\mathbf{P}$ derives from the boundary terms and represents the external virtual work done by the virtual nodal displacements $\delta\mathbf{U}$ on the externally applied nodal forces $\mathbf{P}$. Eq. (14) is the fundamental equation for the displacement-based finite element formulation of the RC frame element with bond slip. The element displacements $\mathbf{u}(x)$ are related to the nodal displacements $\mathbf{U}$ through the displacement shape functions $\mathbf{N}_u^{\mathrm{DB}}(x)$

$$\mathbf{u}(x) = \begin{Bmatrix} \overline{\mathbf{u}}(x) \\ \bar{\bar{\mathbf{u}}}(x) \end{Bmatrix} = \begin{Bmatrix} \mathbf{N}_{\overline{u}}^{\mathrm{DB}}(x) \\ \mathbf{N}_{\bar{\bar{u}}}^{\mathrm{DB}}(x) \end{Bmatrix}\mathbf{U} = \mathbf{N}_u^{\mathrm{DB}}(x)\mathbf{U} \quad (15)$$

where the superscript DB denotes the displacement-based formulation. It follows that $\mathbf{d}_B(x) = \mathbf{B}_B^{\mathrm{DB}}(x)\mathbf{U}$ and $\mathbf{d}_b(x) = \mathbf{B}_b^{\mathrm{DB}}(x)\mathbf{U}$, where $\mathbf{B}_B^{\mathrm{DB}}(x) = \partial_B\mathbf{N}_u^{\mathrm{DB}}(x)$ and $\mathbf{B}_b^{\mathrm{DB}}(x) = \partial_b\mathbf{N}_u^{\mathrm{DB}}(x)$. Upon substitution of Eq. (15) into Eq. (14) and from the arbitrariness of $\delta\mathbf{U}$

$$\int_L \begin{Bmatrix} \mathbf{B}_B^{\mathrm{DB}}(x) \\ \mathbf{B}_b^{\mathrm{DB}}(x) \end{Bmatrix}^T \begin{bmatrix} \mathbf{k}_B^0(x) & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_b^0(x) \end{bmatrix} \begin{Bmatrix} \mathbf{B}_B^{\mathrm{DB}}(x) \\ \mathbf{B}_b^{\mathrm{DB}}(x) \end{Bmatrix} dx\Delta\mathbf{U} = \mathbf{P} - \int_L \mathbf{B}_B^{\mathrm{DB}^T}(x)\mathbf{D}_B^0(x)dx - \int_L \mathbf{B}_b^{\mathrm{DB}^T}(x)\mathbf{D}_b^0(x)dx \quad (16)$$

Eq. (16) is written in the compact form

$$\mathbf{K}^0\Delta\mathbf{U} = \mathbf{P} - \mathbf{Q}^0 \quad (17)$$

where $\mathbf{K}^0$ = the element stiffness matrix and $\mathbf{Q}^0$ contains the element resisting forces

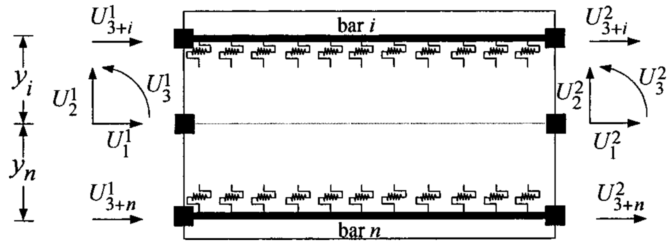

Fig. 5. Two-node displacement-based reinforced concrete element

$$\mathbf{K}^0 = \mathbf{K}_B^0 + \mathbf{K}_b^0 \qquad \mathbf{Q}^0 = \mathbf{Q}_B^0 + \mathbf{Q}_b^0 \quad (18)$$

$\mathbf{K}_B^0$ and $\mathbf{K}_b^0$ are the beam and bond contributions, respectively, to the element stiffness matrix

$$\mathbf{K}_B^0 = \int_L \mathbf{B}_B^{\mathrm{DB}^T}(x)\mathbf{k}_B^0(x)\mathbf{B}_B^{\mathrm{DB}}(x)dx \qquad \mathbf{K}_b^0 = \int_L \mathbf{B}_b^{\mathrm{DB}^T}(x)\mathbf{k}_b^0(x)\mathbf{B}_b^{\mathrm{DB}}(x)dx \quad (19)$$

$\mathbf{Q}_B^0$ = the beam contribution and $\mathbf{Q}_b^0$ = the bond contributions to the element resisting forces

$$\mathbf{Q}_B^0 = \int_L \mathbf{B}_B^{\mathrm{DB}^T}(x)\mathbf{D}_B^0(x)dx \qquad \mathbf{Q}_b^0 = \int_L \mathbf{B}_b^{\mathrm{DB}^T}(x)\mathbf{D}_b^0(x)dx \quad (20)$$

The right-hand side of Eq. (17) is the force residual corresponding to the weak statement of the equilibrium Eq. (5) and vanishes when the equilibrium configuration is reached.

The element used in this paper comprises a two-node beam, plus n two-node bars with bond interfaces (Fig. 5). Higher order elements were also considered during the element development, but the gain in element precision was outweighed by the element complexity and additional computational effort. The two-node element has a linear axial displacement field in the beam and in the bars with bond interfaces, a cubic transverse displacement field in the beam, and quadratic slip fields along the interfaces. These proposed displacement fields satisfy the C^0 and C^1 continuities for the axial and transverse displacements required by the variational indices in Eq. (13). More details on the element shape functions and on the element formulation are given in Spacone and Limkatanyu (2000). The two-node displacement-based element, although numerically stable, is not very accurate and is used primarily as a reference model for the following, more advanced formulations.

Force-Based (Force-Hybrid) Formulation

The force-based formulation stems from the force-based steel–concrete composite beam element with partial interaction proposed by Salari et al. (1998). The force-based formulation is derived from the total complementary potential energy functional and is the dual of the displacement-based formulation. The element internal force fields $\mathbf{D}_B(x)$ and $\mathbf{D}_b(x)$ serve as the primary unknowns and are expressed in terms of the element nodal forces through appropriate force shape functions. The force shape functions are derived such that the equilibrium equations (5) are sat-

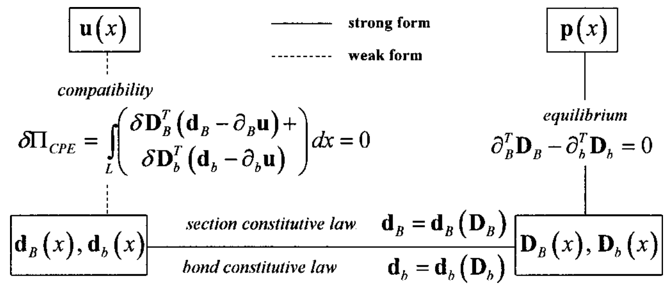

Fig. 6. Tonti's diagram for reinforced concrete beam with bond slip: force-based formulation

isfied pointwise along the element. On the other hand, the beam compatibility equation (6) and the bond compatibility equation (8) are satisfied only in an integral sense. The steps involved in the force-based formulation are schematically represented in Tonti 's diagram (Fig. 6).

The total complementary potential energy functional Π_{CPE} is

$$\Pi_{\mathrm{CPE}}[\mathbf{D}_B(x),\mathbf{D}_b(x)] = \int_L \mathbf{D}_B^T(x)[\mathbf{d}_B(x) - \partial_B\mathbf{u}(x)]dx + \int_L \mathbf{D}_b^T(x)[\mathbf{d}_b^T(x) - \partial_b\mathbf{u}(x)]dx \quad (21)$$

According to the principle of stationary complementary potential energy, the compatible configuration is obtained when Π_{CPE} reaches a stationary value, i.e., when

$$\delta\Pi_{\mathrm{CPE}}[\delta\mathbf{D}_B(x),\delta\mathbf{D}_b(x)] = \int_L \delta\mathbf{D}_B^T(x)[\mathbf{d}_B(x) - \partial_B\mathbf{u}(x)]dx + \int_L \delta\mathbf{D}_b^T(x)[\mathbf{d}_b(x) - \partial_b\mathbf{u}(x)]dx = 0 \quad (22)$$

Upon substitution of the section linearized laws, Eqs. (10) and (22) are written

$$\delta\Pi_{\mathrm{CPE}} = \int_L \delta\mathbf{D}_B^T(x)[\mathbf{d}_B^0(x) + \mathbf{f}_B^0(x)\Delta\mathbf{D}_B(x) - \partial_B\mathbf{u}(x)]dx + \int_L \delta\mathbf{D}_b^T(x)[\mathbf{d}_b^0(x) + \mathbf{f}_b^0(x)\Delta\mathbf{D}_b(x) - \partial_b\mathbf{u}(x)]dx = 0 \quad (23)$$

where $\delta\mathbf{D}_B(x)$ and $\delta\mathbf{D}_b(x)$ = the virtual, equilibrated section and bond-interface force fields, respectively, and $\mathbf{f}_B^0(x)$ and $\mathbf{f}_b^0(x)$ = the initial flexibility matrices of the section and of the bond interfaces, respectively. Integration by parts of Eq. (23) and substitution of the equilibrium equation (5) lead to the following matrix equation:

$$\int_L \begin{Bmatrix} \delta\mathbf{D}_B(x) \\ \delta\mathbf{D}_b(x) \end{Bmatrix}^T \begin{bmatrix} \mathbf{f}_B^0(x) & \mathbf{0} \\ \mathbf{0} & \mathbf{f}_b^0(x) \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{D}_B(x) \\ \Delta\mathbf{D}_b(x) \end{Bmatrix} dx = \delta\overline{\mathbf{Q}}^T\overline{\mathbf{U}} - \int_L \begin{Bmatrix} \delta\mathbf{D}_B(x) \\ \delta\mathbf{D}_b(x) \end{Bmatrix}^T \begin{Bmatrix} \mathbf{d}_B^0(x) \\ \mathbf{d}_b^0(x) \end{Bmatrix} dx \quad (24)$$

where $\delta\overline{\mathbf{Q}}^T\overline{\mathbf{U}}$ = the boundary term and represents the external virtual work done by $\delta\overline{\mathbf{Q}}$ (the virtual element nodal forces without rigid body modes) on $\overline{\mathbf{U}}$ (the corresponding element nodal displacements without rigid body modes). The element is formulated without rigid body modes in view of its implementation in a general-purpose finite element code, which requires inversion of the element flexibility matrix (Spacone et al. 1996). Eq. (24) is the fundamental equation of the force-based finite element formulation. To obtain the discrete form of Eq. (24), the section forces $\mathbf{D}_B(x)$ and the bond-interface forces $\mathbf{D}_b(x)$ are expressed in terms of the element nodal forces without rigid body modes $\overline{\mathbf{Q}}$ and of the bond-interface forces $\overline{\mathbf{Q}}_b$ at selected reference points along the interface, according to the following matrix relation:

$$\begin{Bmatrix} \mathbf{D}_B(x) \\ \mathbf{D}_b(x) \end{Bmatrix} = \begin{bmatrix} \mathbf{N}_{\mathrm{BB}}^{\mathrm{FB}}(x) & \mathbf{N}_{\mathrm{Bb}}^{\mathrm{FB}}(x) \\ \mathbf{N}_{\mathrm{bB}}^{\mathrm{FB}}(x) & \mathbf{N}_{\mathrm{bb}}^{\mathrm{FB}}(x) \end{bmatrix} \begin{Bmatrix} \overline{\mathbf{Q}} \\ \overline{\mathbf{Q}}_b \end{Bmatrix} \quad (25)$$

where the superscript FB denotes the force-based formulation, and $\mathbf{N}_{\mathrm{BB}}^{\mathrm{FB}}(x)$, $\mathbf{N}_{\mathrm{Bb}}^{\mathrm{FB}}(x)$, $\mathbf{N}_{\mathrm{bB}}^{\mathrm{FB}}(x)$, and $\mathbf{N}_{\mathrm{bb}}^{\mathrm{FB}}(x)$ are the force shape functions. Substitution of Eq. (25) into Eq. (24) and from the arbitrariness of $\delta\overline{\mathbf{Q}}$ and $\delta\overline{\mathbf{Q}}_b$, the following matrix expression results:

$$\begin{bmatrix} \overline{\mathbf{F}}_{\mathrm{BB}}^0 & \overline{\mathbf{F}}_{\mathrm{Bb}}^0 \\ \overline{\mathbf{F}}_{\mathrm{Bb}}^{0^T} & \overline{\mathbf{F}}_{\mathrm{bb}}^0 \end{bmatrix} \begin{Bmatrix} \Delta\overline{\mathbf{Q}} \\ \Delta\overline{\mathbf{Q}}_b \end{Bmatrix} = \begin{Bmatrix} \overline{\mathbf{U}} \\ \mathbf{0} \end{Bmatrix} - \begin{Bmatrix} \overline{\mathbf{r}}^0 \\ \overline{\mathbf{r}}_b^0 \end{Bmatrix} \quad (26)$$

where $\overline{\mathbf{F}}_{\mathrm{BB}}^0$, $\overline{\mathbf{F}}_{\mathrm{Bb}}^0$, and $\overline{\mathbf{F}}_{\mathrm{bb}}^0$ are the following flexibility terms:

$$\overline{\mathbf{F}}_{\mathrm{BB}}^0 = \int_L (\mathbf{N}_{\mathrm{BB}}^{\mathrm{FB}^T}\mathbf{f}_B^0\mathbf{N}_{\mathrm{BB}}^{\mathrm{FB}} + \mathbf{N}_{\mathrm{bB}}^{\mathrm{FB}^T}\mathbf{f}_b^0\mathbf{N}_{\mathrm{bB}}^{\mathrm{FB}})dx$$

$$\overline{\mathbf{F}}_{\mathrm{Bb}}^0 = \int_L (\mathbf{N}_{\mathrm{BB}}^{\mathrm{FB}^T}\mathbf{f}_B^0\mathbf{N}_{\mathrm{Bb}}^{\mathrm{FB}} + \mathbf{N}_{\mathrm{bB}}^{\mathrm{FB}^T}\mathbf{f}_b^0\mathbf{N}_{\mathrm{bb}}^{\mathrm{FB}})dx \quad (27)$$

$$\overline{\mathbf{F}}_{\mathrm{bb}}^0 = \int_L (\mathbf{N}_{\mathrm{Bb}}^{\mathrm{FB}^T}\mathbf{f}_B^0\mathbf{N}_{\mathrm{Bb}}^{\mathrm{FB}} + \mathbf{N}_{bb}^{\mathrm{FB}^T}\mathbf{f}_b^0\mathbf{N}_{\mathrm{bb}}^{\mathrm{FB}})dx$$

$\overline{\mathbf{r}}^0$ and $\overline{\mathbf{r}}_b^0$ = the displacements at the element and bond degrees of freedom, respectively, compatible with the internal deformations $\mathbf{d}_B^0$ and $\mathbf{d}_b^0$

$$\overline{\mathbf{r}}^0 = \int_L (\mathbf{N}_{\mathrm{BB}}^{\mathrm{FB}^T}\mathbf{d}_B^0 + \mathbf{N}_{\mathrm{bB}}^{\mathrm{FB}^T}\mathbf{d}_b^0)dx$$

$$\overline{\mathbf{r}}_b^0 = \int_L (\mathbf{N}_{\mathrm{Bb}}^{\mathrm{FB}^T}\mathbf{d}_B^0 + \mathbf{N}_{\mathrm{bb}}^{\mathrm{FB}^T}\mathbf{d}_b^0)dx \quad (28)$$

Similar to the element nodal force residuals in the stiffness equation (17), $\overline{\mathbf{U}} - \overline{\mathbf{r}}^0$ and $-\overline{\mathbf{r}}_b^0$ = the element nodal and bond displacement residuals, respectively, in the flexibility equation (26). The zero term on the right-hand side of Eq. (26) implies that the relative bond slips at the selected reference points along the interfaces are equal to zero. This condition is similar to the known displacement conditions that are used to determine the redundant forces in statically indeterminate structures by the classical force method.

The redundant force unknowns $\Delta\overline{\mathbf{Q}}_b$ are eliminated through static condensation in Eq. (26). The second equation in Eq. (26) yields $\Delta\overline{\mathbf{Q}}_b = -(\overline{\mathbf{F}}_{\mathrm{bb}}^0)^{-1}(\overline{\mathbf{F}}_{\mathrm{bB}}^0\Delta\overline{\mathbf{Q}} + \overline{\mathbf{r}}_b^0)$, which substituted in the first equation yields

$$\overline{\mathbf{F}}^0\Delta\overline{\mathbf{Q}} = \overline{\mathbf{U}} - \overline{\mathbf{U}}_B^0 - \overline{\mathbf{U}}_b^0 \quad (29)$$

where $\overline{\mathbf{F}}$ = the element flexibility matrix defined as

$$\overline{\mathbf{F}}^0 = \overline{\mathbf{F}}_{\mathrm{BB}}^0 - \overline{\mathbf{F}}_{\mathrm{Bb}}^0(\overline{\mathbf{F}}_{\mathrm{bb}}^0)^{-1}\overline{\mathbf{F}}_{\mathrm{bB}}^0 \quad (30)$$

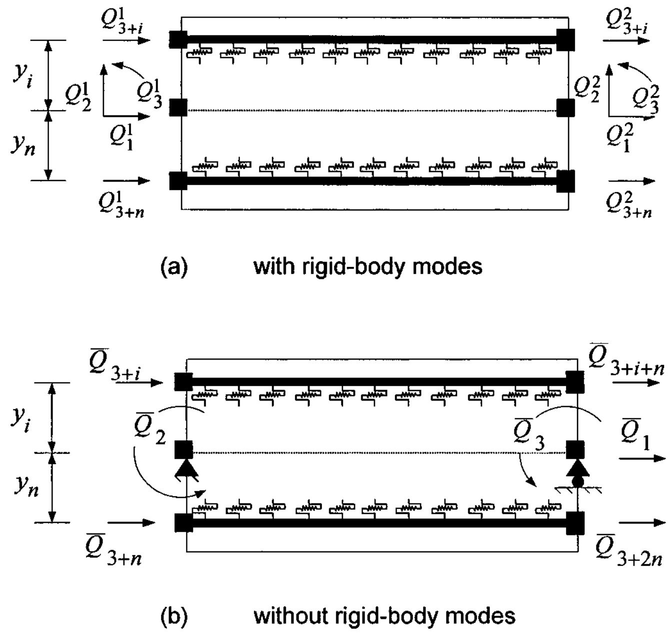

Fig. 7. Two-node force-based reinforced concrete element with bond slip

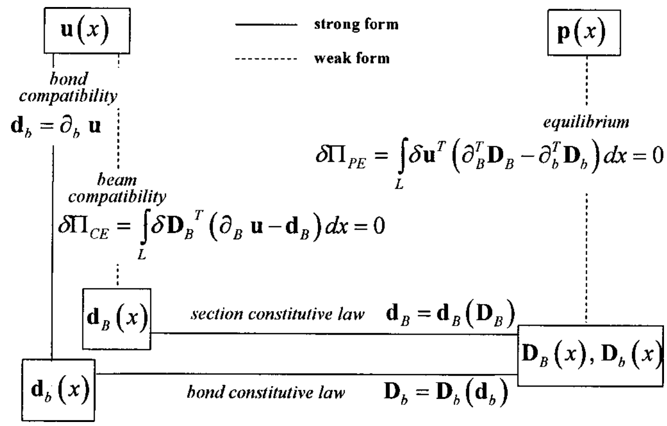

Fig. 8. Tonti's diagram for reinforced concrete beam with bond slip: Hellinger–Reissner mixed formulation

and $\overline{\mathbf{U}}_B^0$ = the contribution of the beam component and $\overline{\mathbf{U}}_b^0$ = the contributions of the bond interfaces, to the element nodal displacements $\overline{\mathbf{U}}^0$ without rigid body modes

$$\overline{\mathbf{U}}_B^0 = \int_L [\mathbf{N}_{\mathrm{BB}}^{\mathrm{FB}^T} - \overline{\mathbf{F}}_{\mathrm{Bb}}^0 (\overline{\mathbf{F}}_{\mathrm{bb}}^0)^{-1} \mathbf{N}_{\mathrm{Bb}}^{\mathrm{FB}^T}] \mathbf{d}_B^0 dx$$
$$\overline{\mathbf{U}}_b^0 = \int_L [\mathbf{N}_{\mathrm{bB}}^{\mathrm{FB}^T} - \overline{\mathbf{F}}_{\mathrm{Bb}}^0 (\overline{\mathbf{F}}_{\mathrm{bb}}^0)^{-1} \mathbf{N}_{\mathrm{bb}}^{\mathrm{FB}^T}] \mathbf{d}_b^0 dx \quad (31)$$

The right-hand side vector $\overline{\mathbf{U}} - \overline{\mathbf{U}}_B^0 - \overline{\mathbf{U}}_b^0$ of Eq. (29) = the element nodal displacement residuals corresponding to the weak form of the compatibility conditions Eqs. (6) and (8) and vanishes when the compatible configuration is reached. In order to implement the force-based element in a general-purpose displacement-based finite element program, it is necessary to introduce a special state-determination procedure (which is discussed in the companion paper: Limkatanyu and Spacone 2002). This procedure computes the element stiffness matrix and the resisting forces for the element without rigid body modes and then adds the rigid body modes back into the element. The resulting hybrid solution procedure, required to interface the force-based element formulation with the displacement-based structural solver, led the authors to rename the element formulation force hybrid.

Fig. 7 shows the two-node force-based RC frame element with and without rigid body modes. Adding the rigid body modes to the element of Fig. 7(b) or filtering out the rigid body modes from the element of Fig. 7(a) is accomplished through matrix transformations based on equilibrium and compatibility between the two systems.

In structures that are internally statically determinate, such as the RC beam with the perfect bond of Spacone et al. (1996), the internal force distributions can be determined exactly from equilibrium. In the RC beam model with bond slip of Fig. 7, which is internally statically indeterminate, the internal force distributions cannot be exactly determined from equilibrium only, except for some special, simple linear–elastic structures. The bond-interface forces serve as the redundant forces in this element. Assumptions on the bond-force distributions are made. This procedure is identical to that followed by Salari et al. (1998) for the steel–concrete composite beam with deformable shear connectors. In the proposed formulation of a RC element with bond slip, the bond-force distributions are assumed to be cubic functions. During the element development, a quadratic bond-interface force distribution was also tested, but it was discarded because it did not accurately represent the actual bond distribution. The beam axial force and bending moment and the bar axial force distributions corresponding to the cubic bond-interface force distributions are quartic functions.

Hellinger–Reissner Mixed Formulation

In the mixed formulation, the section forces $\mathbf{D}_B(x)$ are expressed in terms of the element nodal forces through force shape functions, and the beam displacements $\mathbf{u}(x)$ are written as functions of the element nodal displacements via displacement shape functions. The beam element nodal forces and nodal displacements serve as the primary element unknowns. The equilibrium equation (5) and the beam compatibility equation (6) are satisfied in an integral (or weak) form. On the other hand, compatibility of the bond-interface field is satisfied in a strong form through the compatibility equation (8). The H–R mixed functional Π_{HR} (Felippa 2000) is defined as

$$\Pi_{\mathrm{HR}}[\mathbf{u}(x), \mathbf{D}_B(x)] = \Pi_{\mathrm{EQ}}[\mathbf{u}(x)] + \Pi_{\mathrm{CM}}[\mathbf{D}_B(x)] \quad (32)$$

where Π_{EQ} = the weak form of equilibrium and Π_{CM} = the weak form of the beam compatibility. According to the stationary principle, the compatible equilibrium configuration is obtained when Π_{HR} reaches a stationary value, i.e., when

$$\delta\Pi_{\mathrm{HR}}[\delta\mathbf{u}(x), \delta\mathbf{D}_B(x)] = \delta\Pi_{\mathrm{EQ}}[\delta\mathbf{u}(x)] + \delta\Pi_{\mathrm{CM}}[\delta\mathbf{D}_B(x)] = 0 \quad (33)$$

which implies $\delta\Pi_{\mathrm{EQ}} = 0$ and $\delta\Pi_{\mathrm{CM}} = 0$. The mixed formulation is schematically represented in Tonti's diagram (Fig. 8).

Π_{EQ} is defined by Eq. (11) and its first variation is

$$\delta\Pi_{\mathrm{EQ}}[\delta\mathbf{u}(x)] = \int_L \delta\mathbf{u}^T(x)[\partial_B^T \mathbf{D}_B(x) - \partial_b^T \mathbf{D}_b(x)] dx = 0 \quad (34)$$

where $\delta\mathbf{u}(x)$ = a compatible virtual displacement field. In order to move the differential operators ∂_B and ∂_b from the section forces

$\mathbf{D}_B(x)$ and bond-interface forces $\mathbf{D}_b(x)$ to the virtual displacement field $\delta\mathbf{u}(x)$, integration by parts is applied to Eq. (34)

$$\int_L \{[\partial_B \delta\mathbf{u}(x)]^T \mathbf{D}_B(x) + [\partial_b \delta\mathbf{u}(x)]^T \mathbf{D}_b(x)\} dx = \delta\mathbf{U}^T\mathbf{P} \quad (35)$$

where $\delta\mathbf{U}^T\mathbf{P}$ = the boundary terms and represents the external virtual work done by the element virtual nodal displacements $\delta\mathbf{U}$ on the externally applied nodal forces $\mathbf{P}$. Upon application of the linear incremental form of the material constitutive law Eq. (9) and after imposing the bond-interface compatibility Eq. (8), Eq. (35) is written as

$$\int_L \delta\mathbf{u}^T(x)[\partial_B^T \Delta\mathbf{D}_B(x) + \partial_b^T \mathbf{k}_b^0(x)\partial_b \Delta\mathbf{u}(x)]dx$$

$$= \delta\mathbf{U}^T\mathbf{P} - \int_L \delta\mathbf{u}^T(x)[\partial_B^T \mathbf{D}_B^0(x) + \partial_b^T \mathbf{D}_b^0(x)]dx \quad (36)$$

Eq. (36) is the linear incremental form of the virtual displacement principle. As for the variation of the second term in Eq. (32), $\delta\Pi_{\text{CM}}$ is written as

$$\delta\Pi_{\text{CM}}[\mathbf{D}_B(x)] = \int_L \delta\mathbf{D}_B^T(x)[\partial_B\mathbf{u}(x) - \mathbf{d}_B(x)]dx = 0 \quad (37)$$

where $\delta\mathbf{D}_B(x)$ = a virtual, equilibrated force field that serves as weight function. Upon substitution of Eq. (10) and after expressing the displacement field in its linearized form $\mathbf{u}(x) = \mathbf{u}^0(x) + \Delta\mathbf{u}(x)$, Eq. (37) is written

$$\int_L \delta\mathbf{D}_B^T(x)[\partial_B \Delta\mathbf{u}(x) - \mathbf{f}_B^0(x)\Delta\mathbf{D}_B(x)]dx$$

$$= \int_L \delta\mathbf{D}_B^T(x)[\mathbf{d}_B^0(x) - \partial_B\mathbf{u}^0(x)]dx \quad (38)$$

Eqs. (36) and (38) form the backbone of the H–R mixed finite element formulation for the RC frame element with bond slip. They are combined into the following form:

$$\int_L \begin{Bmatrix} \delta\mathbf{u}(x) \\ \delta\mathbf{D}_B(x) \end{Bmatrix}^T \begin{bmatrix} \partial_b^T \mathbf{k}_b^0(x)\partial_b & \partial_B^T \\ \partial_B & -\mathbf{f}_B^0(x) \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{u}(x) \\ \Delta\mathbf{D}_B(x) \end{Bmatrix} dx$$

$$= \delta\mathbf{U}^T\mathbf{P} - \int_L \begin{Bmatrix} \delta\mathbf{u}(x) \\ \delta\mathbf{D}_B(x) \end{Bmatrix}^T \begin{Bmatrix} \partial_B^T\mathbf{D}_B^0(x) + \partial_b^T\mathbf{D}_b^0(x) \\ \partial_B\mathbf{u}^0(x) - \mathbf{d}_B^0(x) \end{Bmatrix} dx \quad (39)$$

To obtain the discrete form of Eq. (39), the beam force and displacement fields are expressed in terms of the nodal force degrees of freedom $\mathbf{Q}_R$ (to be specified later in the paper) and the nodal displacements $\mathbf{U}$ through the force and displacement shape functions, respectively,

$$\mathbf{u}(x) = \begin{Bmatrix} \bar{\mathbf{u}}(x) \\ \bar{\bar{\mathbf{u}}}(x) \end{Bmatrix} = \begin{Bmatrix} \mathbf{N}_{\bar{u}}^{\text{H-R}}(x) \\ \mathbf{N}_{\bar{\bar{u}}}^{\text{H-R}}(x) \end{Bmatrix} \mathbf{U} = \mathbf{N}_u^{\text{H-R}}(x)\mathbf{U}$$

$$\mathbf{D}_B(x) = \begin{Bmatrix} \bar{\mathbf{D}}(x) \\ \bar{\bar{\mathbf{D}}}(x) \end{Bmatrix} = \begin{Bmatrix} \mathbf{N}_{\bar{F}}^{\text{H-R}}(x) \\ \mathbf{N}_{\bar{\bar{F}}}^{\text{H-R}}(x) \end{Bmatrix} \mathbf{Q}_R = \mathbf{N}_F^{\text{H-R}}(x)\mathbf{Q}_R \quad (40)$$

where superscript H–R = the H–R mixed formulation. $\mathbf{N}_{\bar{u}}^{\text{H-R}}(x)$ and $\mathbf{N}_{\bar{\bar{u}}}^{\text{H-R}}(x)$ contain the displacement shape functions for the beam and bar displacements, respectively. $\mathbf{N}_{\bar{F}}^{\text{H-R}}(x)$ and $\mathbf{N}_{\bar{\bar{F}}}^{\text{H-R}}(x)$ contain the force shape functions for the beam and bar forces, respectively. Upon substituting Eq. (40) into Eq. (39) and from the arbitrariness of $\delta\mathbf{Q}_R$ and $\delta\mathbf{U}$ the following equation results:

$$\begin{bmatrix} \mathbf{K}_b^0 & \mathbf{T}^T \\ \mathbf{T} & -\bar{\mathbf{F}}_B^0 \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{U} \\ \Delta\mathbf{Q}_R \end{Bmatrix} = \begin{Bmatrix} \mathbf{P} - \mathbf{T}^T\mathbf{Q}_R^0 - \mathbf{Q}_b^0 \\ \bar{\mathbf{U}}_{Rr}^0 \end{Bmatrix} \quad (41)$$

where $\mathbf{K}_b^0$ = the contribution of the bond interfaces to the element stiffness matrix, defined as

$$\mathbf{K}_b^0 = \int_L \mathbf{B}_b^{\text{H-R}^T}(x)\mathbf{k}_b^0(x)\mathbf{B}_b^{\text{H-R}}(x)dx \quad (42)$$

with $\mathbf{B}_B^{\text{H-R}}(x) = \partial_B\mathbf{N}_u^{\text{H-R}}(x)$, $\mathbf{B}_b^{\text{H-R}}(x) = \partial_b\mathbf{N}_u^{\text{H-R}}(x)$. $\bar{\mathbf{F}}_B^0$ is the beam flexibility matrix, defined as

$$\bar{\mathbf{F}}_B^0 = \int_L \mathbf{N}_F^{\text{H-R}^T}(x)\mathbf{f}_B^0(x)\mathbf{N}_F^{\text{H-R}}(x)dx \quad (43)$$

Matrix $\mathbf{T}$ serves as the transformation matrix between the force degrees of freedom and the displacement degrees of freedom, and is defined as

$$\mathbf{T} = \int_L \mathbf{N}_F^{\text{H-R}^T}(x)\mathbf{B}_B^{\text{H-R}}(x)dx \quad (44)$$

$\bar{\mathbf{U}}_{Rr}^0$ contains the beam nodal displacement residuals and is defined as

$$\bar{\mathbf{U}}_{Rr}^0 = \int_L \mathbf{N}_F^{\text{H-R}^T}(x)\mathbf{d}_B^0(x) - \mathbf{T}\mathbf{U}^0 \quad (45)$$

$\mathbf{Q}_b^0$ = the contribution of the bond interfaces to the element forces

$$\mathbf{Q}_b^0 = \int_L \mathbf{B}_b^{\text{H-R}^T}(x)\mathbf{D}_b^0(x)dx \quad (46)$$

The right-hand side of Eq. (41) contains the force and displacement residuals, which correspond to the weak forms of equilibrium and compatibility, respectively.

In view of the element implementation into a general-purpose finite element program, the force degrees of freedoms in Eq. (41) are eliminated using static condensation. Consequently, the force continuity between adjacent elements is locally relaxed. From the second equation in Eq. (41) the beam element nodal forces are computed as $\Delta\mathbf{Q}_R = (\bar{\mathbf{F}}_B^0)^{-1}(\mathbf{T}\Delta\mathbf{U} - \bar{\mathbf{U}}_{Rr}^0)$ and they are substituted into the first equation in Eq. (41) to obtain

$$(\mathbf{T}^T(\bar{\mathbf{F}}_B^0)^{-1}\mathbf{T} + \mathbf{K}_b^0)\Delta\mathbf{U} = \mathbf{P} - \mathbf{T}^T\mathbf{Q}_R^0 - \mathbf{Q}_b^0 + \mathbf{T}^T(\bar{\mathbf{F}}_B^0)^{-1}\bar{\mathbf{U}}_{Rr}^0 \quad (47)$$

where the term $\mathbf{T}^T\mathbf{Q}_R^0 + \mathbf{Q}_b^0 - \mathbf{T}^T(\bar{\mathbf{F}}_B^0)^{-1}\bar{\mathbf{U}}_{Rr}^0$ represents the element forces. In Eq. (47), $\mathbf{Q}_b^0$ are forces directly dual to the nodal displacements $\mathbf{U}$, while $\mathbf{Q}_R^0$ and $(\bar{\mathbf{F}}_B^0)^{-1}\bar{\mathbf{U}}_{Rr}^0$ represent the nodal forces at the force degrees of freedom. These latter are premultiplied by the transformation matrix $\mathbf{T}$ to become dual to the nodal displacements $\mathbf{U}$.

In order to ensure numerical stability of the solution, the element degrees of freedom and the element displacement and force shape functions cannot be arbitrarily selected for mixed elements, as discussed in Zienkiewicz and Taylor (1989). For the extreme case when $\mathbf{K}_b^0$ in Eq. (47) vanishes, the element stiffness matrix $\mathbf{T}^T(\bar{\mathbf{F}}_B^0)^{-1}\mathbf{T}$ should have the correct number of zero eigenvalues corresponding to the number of rigid body modes to ensure stability of the formulation. This condition is satisfied if the rank of the element stiffness matrix is not smaller than the number of displacement degrees of freedom minus the number of rigid body modes. The rank of this term cannot be greater than the rank of $(\bar{\mathbf{F}}_B^0)^{-1}$. In other words, the number of force degrees of freedom

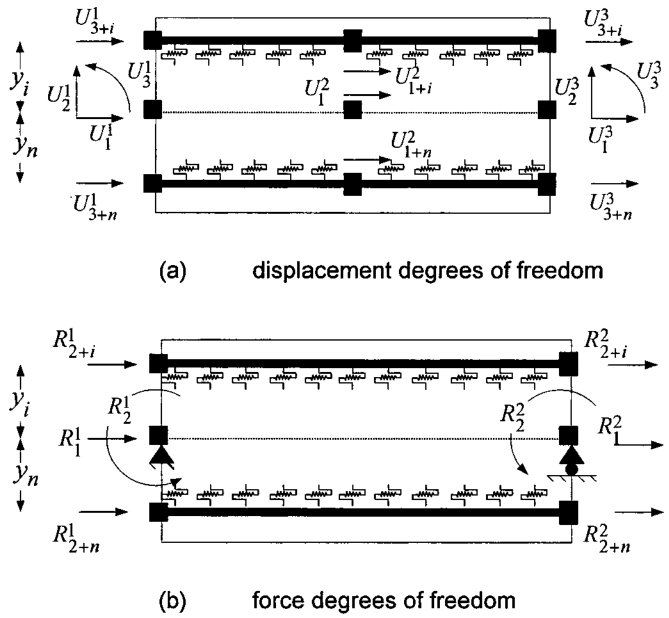

Fig. 9. Mixed reinforced concrete element with bond slip

$n_{\mathbf{R}}$ cannot be smaller than the number of displacement degrees of freedom without rigid body modes $n_{\bar{\mathbf{U}}}$ (Ayoub and Filippou 2000)

$$n_{\mathbf{R}} \geqslant n_{\bar{\mathbf{U}}} \tag{48}$$

The above condition is necessary but not sufficient to ensure element convergence.

The degree of the displacement shape functions is selected after considering the variational indices in Eq. (37). Because the first derivative of the axial displacement and the second derivative of the transverse displacement appear in Eq. (37), C^0 and C^1 continuous functions are selected for the axial and transverse displacements, respectively. As a result, a cubic Hermitian polynomial is used for the beam transverse displacement and quadratic polynomials are used for the beam and bar axial displacements.

The selection of the degree of the force shape functions considers de Veubeke's principle of limitation (de Veubeke 1965), which states that there is no further improvement in accuracy obtained by increasing the degree of the force fields beyond the degree of their corresponding deformation fields. The above limitation is only pertinent to the linear elastic case. Higher order force shape functions may possibly improve the element performance in the inelastic range but they were not considered in the present study in order to minimize the computational cost of the element. Because the cubic transverse displacement implies a linear curvature variation and the quadratic axial displacement implies a linear axial strain variation, linear bending moment and axial force variations were selected for the force shape functions.

The displacement degrees of freedom that derive from the above considerations are shown in Fig. 9(a). The displacement degrees of freedom use a three-node beam with n three-node bars. Each end node has $3+n$ degrees of freedoms, three for the beam and one axial displacement for each of the n bars with bond interfaces. The middle node has only $1+n$ degrees of freedoms, one axial displacement in the beam and one axial displacement for each of the n bars with bond interfaces. As a result, the axial displacement distributions in the beam and in the bars with bond interfaces are quadratic; the transverse displacement distribution in the beam is cubic and the distribution of bond slip is quadratic.

The combination of a two-node beam with n two-node bars with bond interfaces is used to define the element force degrees of freedoms without rigid body modes [Fig. 9(b)]. Each node has $2+n$ degrees of freedoms, one for the axial force and one for the bending moment in the beam, plus one axial force for each of the n bars with bond interfaces. The nodal force components R in Fig. 9(b) are grouped in the beam nodal force vector $\mathbf{Q}_R$. Consequently, the beam axial force, the beam bending moment, and the bar axial forces are all linear.

Summary

This is the first of two papers proposing three formulations for a RC frame element with bond slip in the rebars. The main objective of this paper is to present the derivation of the governing differential equations (strong form) and to derive three different finite element formulations (weak forms) for the RC frame element with bond interfaces. The three finite elements are based on the displacement-based, the force-based (hybrid), and the Hellinger–Reissner mixed formulations. The displacement-based element is derived from the principle of stationary potential energy functional and uses displacement-shape functions to express the beam and steel bar displacement fields in term of nodal displacement degrees of freedom. The force-based (hybrid) element is derived from the principle of stationary total complementary potential energy functional and employs force-shape functions to express the internal force fields in terms of force degrees of freedom. In this element, the element bond forces at selected reference points serve as internal redundant forces, and are statically condensed out in order to implement the element into a general-propose displacement-based finite element program. As a result, the bond force continuity between adjacent elements is locally relaxed. The mixed element is derived from the principle of stationary Hellinger–Reissner mixed functional. Force shape functions are used to express the beam internal force fields in terms of the beam force degrees of freedom and displacement shape functions are used to express the bond-slip fields in terms of the element displacement degrees of freedom. In the mixed element, the beam force degrees of freedom are statically condensed out in order to implement the element into a general-purpose displacement-based finite element program. Thus, the beam force continuity between adjacent elements is locally relaxed. The stability requirements on the order and continuity of the displacement and force shape functions for the mixed element are established. Finally, Tonti's diagrams are used to schematically represent the set of basic governing equations for both strong and weak forms. The companion paper by Limkatanyu and Spacone (2001) discusses the implementation issues of the three different formulations in a general purpose nonlinear structural analysis software and presents a set of applications and parametric studies.

Acknowledgments

This study was supported by the Royal Thai Fellowship and by Grant No. CMS-9804613 from the National Science Foundation. This support is gratefully acknowledged. Any opinions expressed in this paper are those of the writers and do not reflect the views of the sponsoring agencies. Special thanks go to Dr. Mohammed R. Salari for his assistance with formulations of beams with bond slip and to Professor Carlos A. Felippa of the Aerospace Dept. of the University of Colorado, Boulder for his continuous support on theoretical issues.

Notation

The following symbols are used in this paper:

$\mathbf{B}_B^{\mathrm{DB}}$, $\mathbf{B}_b^{\mathrm{DB}}$, = beam and bond deformation-displacement matrices for the displacement-based formulation;
$\mathbf{B}_B^{\mathrm{H-R}}$, $\mathbf{B}_b^{\mathrm{H-R}}$ = beam and bond deformation-displacement matrices, respectively, for H–R mixed formulation;
$\mathbf{D}_B$, $\overline{\mathbf{D}}$, $\bar{\bar{\mathbf{D}}}$, $\mathbf{D}_b$ = section, beam, bar, and bond force arrays, respectively;
D_{bi} = bond interface force in i bar;
$\mathbf{d}_B$, $\overline{\mathbf{d}}$, $\bar{\bar{\mathbf{d}}}$, $\mathbf{d}_b$ = section, beam, bar, and bond deformation arrays, respectively;
$\overline{\mathbf{F}}_B$ = beam element flexibility matrix (H–R formulation);
$\overline{\mathbf{F}}$ = element flexibility matrix (force-based formulation);
$\overline{\mathbf{F}}_{\mathrm{BB}}$, $\overline{\mathbf{F}}_{\mathrm{bb}}$, $\overline{\mathbf{F}}_{\mathrm{bB}}$ = beam-bond flexibility matrices;
$\mathbf{f}_B$, $\mathbf{f}_b$ = section and bond flexibility matrices;
$\mathbf{K}$, $\mathbf{K}_B$, $\mathbf{K}_b$ = element stiffness matrix, beam, and bond contributions to $\mathbf{K}$;
$\mathbf{k}_B$, $\mathbf{k}_b$ = section and bond stiffness matrices;
M_B = beam section bending moment;
$\mathbf{N}_u^{\mathrm{DB}}$, $\mathbf{N}_{\bar{u}}^{\mathrm{DB}}$, $\mathbf{N}_{\bar{\bar{u}}}^{\mathrm{DB}}$ = displacement shape functions in displacement-based formulation;
$\mathbf{N}_u^{\mathrm{H-R}}$, $\mathbf{N}_{\bar{u}}^{\mathrm{H-R}}$, $\mathbf{N}_{\bar{\bar{u}}}^{\mathrm{H-R}}$ = displacement shape functions in H–R mixed formulation;
$\mathbf{N}_{\mathrm{BB}}^{\mathrm{FB}}$, $\mathbf{N}_{\mathrm{Bb}}^{\mathrm{FB}}$, $\mathbf{N}_{\mathrm{bB}}^{\mathrm{FB}}$, $\mathbf{N}_{\mathrm{bb}}^{\mathrm{FB}}$ = force shape function matrices in force-based formulation;
$\mathbf{N}_F^{\mathrm{H-R}}$, $\mathbf{N}_{\bar{F}}^{\mathrm{H-R}}$, $\mathbf{N}_{\bar{\bar{F}}}^{\mathrm{FH-R}}$ = force shape function matrix for H–R mixed formulation;
N_B, N_i = beam and i bar axial forces;
$n_{\mathbf{R}}$, $n_{\overline{\mathbf{U}}}$ = number of unknown forces and displacements (without rigid body modes);
$\mathbf{P}$ = element external nodal force array;
$\mathbf{p}$ = element force array;
p_y = transverse distributed load;
$\mathbf{Q}$, $\mathbf{Q}_B$, $\mathbf{Q}_b$ = element internal nodal force array, beam, and bond contributions to $\mathbf{Q}$;
$\overline{\mathbf{Q}}$ = element nodal force array without rigid modes;
$\overline{\mathbf{Q}}_b$ = reference bond force array;
$\mathbf{Q}_R$ = beam element force array at force degrees of freedom in H–R formulation;
$\overline{\mathbf{r}}$, $\overline{\mathbf{r}}_b$ = element nodal and bond-interface deformation arrays;
$\mathbf{T}$ = transformation matrix;
$\mathbf{U}$, $\overline{\mathbf{U}}$ = element displacement array, with and without rigid body modes;
$\overline{\mathbf{U}}_B$, $\overline{\mathbf{U}}_b$ = beam and bond contributions to $\overline{\mathbf{U}}$;
$\overline{\mathbf{U}}_{Rr}$ = beam element displacement residual array;
$\mathbf{u}$, $\overline{\mathbf{u}}$, $\bar{\bar{\mathbf{u}}}$ = element section, beam, and bar displacement arrays;
u_B, v_B = beam axial and transverse displacements;
u_i, u_{bi} = axial displacement and relative slip, respectively, in i bar;
$V_B(x)$ = beam section shear force;
y_i = distance of i bar from section reference axis;

$$\partial_B = \begin{bmatrix} \frac{d}{dx} & 0 & \vdots & 0 & \mathbf{0} & 0 \\ 0 & \frac{d^2}{dx^2} & \vdots & 0 & \mathbf{0} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & \frac{d}{dx} & \cdots & 0 \\ \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & 0 & \vdots & 0 & \cdots & \frac{d}{dx} \end{bmatrix}$$

= beam differential operator;

$$\partial_b = \begin{bmatrix} -1 & y_1\frac{d}{dx} & 1 & \mathbf{0} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & y_n\frac{d}{dx} & 0 & \mathbf{0} & 1 \end{bmatrix}$$

= bond differential operator;

ε_B, ε_i = beam and i bar axial strain;
κ_B = beam curvature;
Π_{HR}, Π_{EQ}, Π_{CM} = H–R mixed functional, equilibrium and compatibility contributions to Π_{HR}; and
Π_{PE}, Π_{CPE} = total and complementary potential energy functionals.

References

Ayoub, A., and Filippou, F. C. (2000). "Mixed formulation of nonlinear steel-concrete composite beam element." *J. Struct. Eng.*, 126(3), 371–381.

de Veubeke, B. F. (1965). *Stress analysis*, O. C. Zienkiewicz and G. S. Holister, eds., Wiley, London, 145–197.

Eligehausen, R., Popov, E. P., and Bertero, V. V. (1983). "Local bond stress-slip relationships of deformed bars under generalized excitations: experimental results and analytical model." *Rep. EERC 83-23*, Earthquake Engineering Research Center, Univ. of California, Berkeley, Calif.

Felippa, C. A. (2000). *ASEN 5367 Course Notes: Advanced Finite Element Methods*, Dept. of Aerospace Engineering, Univ. of Colorado, Boulder, http://caswww.colorado.edu/courses.d/AFEM.d/Home.html

Filippou, F. C., D'Ambrisi, A., and Issa, A. (1999). "Effects of reinforcement slip on hysteretic behavior of reinforced concrete frame members." *ACI Struct. J.*, 96(3), 327–335.

Filippou, F. C., and Issa, A. (1988). "Nonlinear analysis of reinforced concrete frames under cyclic load reversals." *Rep. EERC 88/12*, Earthquake Engineering Research Center, Univ. of California, Berkeley, Calif.

Kent, D. C., and Park, R. (1971). "Flexural members with confined Concrete." *J. Struct. Div. ASCE*, 97(ST7), 1964–1990.

Kwak, H. G., and Filippou, F. C. (1995). "A new reinforcing steel model with bond-slip." *Struct. Eng. Mech., Int. J.*, 3(4), 299–312.

Limkatanyu, S., and Spacone, E. (2002). ''Reinforced concrete frame element with bond interfaces. II: State determinations and numerical validation.'' *J. Struct. Eng.* , 128(3), 356–364.

Lowes, L. N. (1999). ''Finite element modeling of reinforced concrete beam-column bridge connections.'' PhD thesis, Univ. of California, Berkeley, Calif.

Menegotto, M., and Pinto, P. E. (1973). ''Method of analysis for cyclically loaded reinforced concrete plane frames including changes in geometry and nonelastic behavior of elements under combined normal force and bending.'' *Proc., IABSE Symp. on Resistance and Ultimate Deformability of Structures Acted on by Well-Defined Repeated Loads*, Lisbon, Portugal, IABSE Edition, Switzerland, 112–123.

Monti, G., Filippou, F. C., and Spacone, E. (1997). ''Finite element for anchored bars under cyclic load reversals.'' *J. Struct. Eng.* 123(5), 614–623.

Monti, G., and Spacone, E. (2000). ''Reinforced concrete fiber beam element with bond-slip.'' *J. Struct. Eng.* 126(6), 654–661.

Ngo, D., and Scordelis, A. C. (1967). ''Finite element analysis of reinforced concrete beams.'' *ACI Struct. J.,* 64(3), 152–163.

Nilson, A. H. (1971). ''Internal measurement of bond-slip.'' *ACI Struct. J.,* 69(7), 439–441.

Rubiano-Benavides, N. R. (1998). ''Predictions of the inelastic seismic response of concrete structures including shear deformations and anchorage slip.'' PhD thesis, Univ. of Texas, Austin, Tex.

Salari, M. R., Spacone, E., Shing, P. B., and Frangopol, D. M. (1998). ''Nonlinear analysis of composite beams with deformable shear connectors.'' *Ground Water,* 124(10), 1148–1158.

Spacone, E., Filippou, F. C., and Taucer, F. F. (1996). ''Fibre beam-column model for nonlinear analysis of R/C frames. Part I: formulation.'' *Earthquake Eng. Struct. Dyn.*, 25, 711–725.

Spacone, E., and Limkatanyu, S. (2000). ''Response of reinforced concrete members including bond-slip effects.'' *ACI Struct. J.,* 97(6), 831–839.

Zienkiewicz, O. C., and Taylor, R. L. (1989). *The finite element method*, 4th Ed., Vol. 1, McGraw-Hill, London.